

International Journal of Geometric Methods in Modern Physics
© World Scientific Publishing Company

THE KERR-SCHILD ANSATZ REVISED

DONATO BINI

*Istituto per le Applicazioni del Calcolo "M. Picone," CNR I-00161 Rome, Italy
ICRA, University of Rome "La Sapienza," I-00185 Rome, Italy
ICRANet, I-65100 Pescara, Italy
binid@icra.it*

ANDREA GERALICO

*Physics Department, University of Rome "La Sapienza," I-00185 Rome, Italy
ICRA, University of Rome "La Sapienza," I-00185 Rome, Italy
ICRANet, I-65100 Pescara, Italy
geralico@icra.it*

ROY P. KERR

*University of Canterbury, Christchurch, New Zealand
ICRANet, I-65100 Pescara, Italy
roy.kerr@canterbury.ac.nz*

Received (Day Month Year)

Revised (Day Month Year)

Kerr-Schild metrics have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field, say k , multiplied by some scalar function, say H . The basic assumption which led to Kerr solution was that k be both geodesic and shearfree. This condition is relaxed here and Kerr-Schild ansatz is revised by treating Kerr-Schild metrics as *exact linear perturbations* of Minkowski spacetime. The scalar function H is taken as the perturbing function, so that Einstein's field equations are solved order by order in powers of H . It turns out that the congruence must be geodesic and shearfree as a consequence of third and second order equations, leading to an alternative derivation of Kerr solution.

Keywords: Kerr-Schild metrics

1. Introduction

Kerr-Schild metrics have the form [1,2]

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \equiv (\eta_{\alpha\beta} - 2H k_\alpha k_\beta) dx^\alpha dx^\beta, \quad (1)$$

where $\eta_{\alpha\beta}$ is the metric for Minkowski space and k_α is a null vector

$$\eta_{\alpha\beta} k^\alpha k^\beta = g_{\alpha\beta} k^\alpha k^\beta = 0, \quad k^\alpha = \eta^{\alpha\beta} k_\beta = g^{\alpha\beta} k_\beta. \quad (2)$$

The inverse metric is also linear in H

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2H k^\alpha k^\beta, \quad (3)$$

2 Donato Bini, Andrea Geralico and Roy P. Kerr

and so the determinant of the metric is independent of H

$$(\eta_{\alpha\gamma} - 2Hk_\alpha k_\gamma)(\eta^{\gamma\beta} + 2Hk^\gamma k^\beta) = \delta_\alpha^\beta \quad \longrightarrow \quad |g_{\alpha\beta}| = |\eta_{\alpha\beta}|. \quad (4)$$

Within this class of general metrics the Kerr solution was obtained in 1963 by a systematic study of algebraically special vacuum solutions [3]. If $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ are the standard Cartesian coordinates for Minkowski spacetime with $\eta_{\alpha\beta} = \text{diag}[-1, 1, 1, 1]$, then for Kerr metric we have

$$-k_\alpha dx^\alpha = dt + \frac{(rx + ay)dx + (ry - ax)dy}{r^2 + a^2} + \frac{z}{r}dz, \quad (5)$$

where r and H are defined implicitly by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \quad H = -\frac{\mathcal{M}r^3}{r^4 + a^2 z^2}. \quad (6)$$

Kerr solution is asymptotically flat and the constants \mathcal{M} and a are the total mass and specific angular momentum for a localized source. They both have the dimension of a length in geometrized units. The vector \mathbf{k} is geodesic and shearfree, implying that Kerr metric is algebraically special according to the Goldberg-Sachs theorem [4]. Moreover, \mathbf{k} is independent of \mathcal{M} and hence a function of a alone. Note that the mass parameter \mathcal{M} appears linearly in the metric, i.e. in H .

In this paper we consider Kerr-Schild metrics (1) as *exact linear perturbations* of Minkowski space and solve Einstein's field equations order by order in powers of H . The results of this analysis will be that \mathbf{k} must be geodesic and shearfree as a consequence of third and second order equations, leading to an alternative derivation of Kerr solution.

2. Modified ansatz

Let ϵ be an arbitrary constant parameter, eventually to be set equal to 1, so that the Kerr-Schild metric (1) reads

$$g_{\alpha\beta} = \eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta, \quad (1)$$

with inverse

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2\epsilon H k^\alpha k^\beta, \quad (2)$$

and suppose that coordinates are chosen so that the components $\eta_{\alpha\beta}$ are constants, but not necessarily of the form $\eta_{\alpha\beta} = \text{diag}[-1, 1, 1, 1]$. The connection is then quadratic in ϵ

$$\Gamma^\gamma_{\alpha\beta} = \epsilon \Gamma_1^\gamma_{\alpha\beta} + \epsilon^2 \Gamma_2^\gamma_{\alpha\beta}, \quad (3)$$

where

$$\begin{aligned} \Gamma_1^\gamma_{\alpha\beta} &= -(Hk_\alpha k^\gamma)_{,\beta} - (Hk_\beta k^\gamma)_{,\alpha} + (Hk_\alpha k_\beta)_{,\lambda} \eta^{\lambda\gamma}, \\ \Gamma_2^\gamma_{\alpha\beta} &= 2H[H(\dot{k}_\alpha k_\beta + \dot{k}_\beta k_\alpha) + \dot{H}k_\alpha k_\beta]k^\gamma \equiv 2Hk^\gamma (Hk_\alpha k^\beta)' \quad , \end{aligned} \quad (4)$$

a “dot” denoting differentiation in the \mathbf{k} direction, i.e. $\dot{f} = \mathbf{k}(f) = f_{,\alpha}k^\alpha$. Note that only the indices of \mathbf{k} can be raised and lowered with the Minkowski metric. Hereafter we will use an “index” 0 to denote contraction with \mathbf{k} , i.e.

$$\begin{aligned}\Gamma^0_{\alpha\beta} &= \Gamma^\gamma_{\alpha\beta}k_\gamma = -\epsilon(Hk_\alpha k_\beta)^\cdot, \\ \Gamma^\gamma_{\alpha 0} &= \Gamma^\gamma_{\alpha\beta}k^\beta = -\epsilon(Hk_\alpha k^\gamma)^\cdot, \\ \Gamma^\gamma_{00} &= \Gamma^\gamma_{\alpha\beta}k^\alpha k^\beta = 0, \\ \Gamma^0_{\alpha 0} &= \Gamma^\gamma_{\alpha\beta}k^\beta k_\gamma = 0.\end{aligned}\tag{5}$$

The determinant of the full metric is independent of ϵ

$$|g_{\alpha\beta}| = |\eta_{\alpha\beta} - 2\epsilon Hk_\alpha k_\beta| = |\eta_{\alpha\beta}| = \text{const.} \quad \longrightarrow \quad \Gamma^\beta_{\alpha\beta} = 0, \tag{6}$$

and the contracted Riemann tensor therefore reduces to

$$R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta} = \Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma}. \tag{7}$$

The simplest component is

$$\begin{aligned}R_{\alpha\beta}k^\alpha k^\beta &= \Gamma^\gamma_{\alpha\beta,\gamma}k^\alpha k^\beta - \Gamma^\gamma_{\delta 0}\Gamma^\delta_{\gamma 0} = \Gamma^\gamma_{00,\gamma} - 2\Gamma^\gamma_{\alpha 0}k^\alpha_{,\gamma} \\ &= 2\epsilon H||\dot{\mathbf{k}}||^2.\end{aligned}\tag{8}$$

In vacuum the LHS is zero, then $||\dot{\mathbf{k}}|| = 0$ and so $\dot{\mathbf{k}}$ is a null-vector orthogonal to another null-vector, \mathbf{k} . Hence $\dot{\mathbf{k}}$ must be parallel to \mathbf{k} and therefore \mathbf{k} is a geodesic vector.

The Ricci tensor expanded as series in ϵ is given by

$$R_{\alpha\beta} = \epsilon R_{1\alpha\beta} + \epsilon^2 R_{2\alpha\beta} + \epsilon^3 R_{3\alpha\beta} + \epsilon^4 R_{4\alpha\beta}. \tag{9}$$

The vacuum Einstein's equations $R_{\alpha\beta} = 0$ imply that contributions of all orders must vanish. Let us evaluate all such components.

The highest components of the expansion for the Ricci tensor are

$$R_{4\alpha\beta} = -\Gamma_2^\rho_{\alpha\sigma}\Gamma_2^\sigma_{\beta\rho} = 0, \tag{10}$$

$$R_{3\alpha\beta} = -\Gamma_1^\rho_{\alpha\sigma}\Gamma_2^\sigma_{\beta\rho} - \Gamma_2^\rho_{\alpha\sigma}\Gamma_1^\sigma_{\beta\rho} = 4H^3||\dot{\mathbf{k}}||^2 k_\alpha k_\beta. \tag{11}$$

The next component of $R_{\alpha\beta}$ is

$$\begin{aligned}R_{2\alpha\beta} &= \Gamma_2^\rho_{\alpha\beta,\rho} - \Gamma_1^\rho_{\alpha\sigma}\Gamma_1^\sigma_{\beta\rho} \\ &= 2H \left[(Hk_\alpha k_\beta)^\cdot + k^\sigma_{,\sigma}(Hk_\alpha k_\beta)^\cdot - H\dot{k}_\alpha \dot{k}_\beta \right] \\ &\quad - H^2\Phi k_\alpha k_\beta - 2Hk_{(\alpha}\psi_{\beta)},\end{aligned}\tag{12}$$

where

$$\Phi = 4\eta^{\gamma\lambda}\eta^{\delta\mu}k_{[\lambda,\delta]}k_{[\mu,\gamma]}, \quad \psi_\alpha = 2\dot{k}^\gamma(Hk_\alpha)_{,\gamma}. \tag{13}$$

Finally, the first component of the Ricci tensor expansion is

$$\begin{aligned}R_{1\alpha\beta} &= \Gamma_1^\gamma_{\alpha\beta,\gamma} \\ &= Ak_\alpha k_\beta + 2k_{(\alpha}B_{\beta)} + X_{\alpha\beta},\end{aligned}\tag{14}$$

4 Donato Bini, Andrea Geralico and Roy P. Kerr

where

$$\begin{aligned}
A &= \eta^{\lambda\gamma} H_{,\lambda\gamma} , \\
B_\beta &= -(Hk^\gamma)_{,\gamma\beta} + \frac{1}{H} \eta^{\lambda\gamma} (H^2 k_{\beta,\gamma})_{,\lambda} , \\
X_{\alpha\beta} &= -2H [(k_{(\alpha,\beta)} k^\gamma)_{,\gamma} + k_{(\alpha,|\gamma|} k^\gamma_{\beta)} - \eta^{\lambda\gamma} k_{\alpha,\gamma} k_{\beta,\lambda}] \\
&\quad - 2k^\gamma [H_{,(\alpha} k_{\beta),\gamma} + H_{,\gamma} k_{(\alpha,\beta)}] \\
&= -2H [\dot{k}_{(\alpha,\beta)} + k^\gamma_{,\gamma} k_{(\alpha,\beta)} - \eta^{\lambda\gamma} k_{\alpha,\gamma} k_{\beta,\lambda}] \\
&\quad - 2\dot{H} k_{(\alpha,\beta)} - 2H_{,(\alpha} \dot{k}_{\beta)} .
\end{aligned} \tag{15}$$

2.1. Kinematical properties of the congruence \mathbf{k}

Taking the covariant derivative of \mathbf{k} gives

$$\nabla_\alpha k_\beta = k_{\beta,\alpha} - \epsilon(Hk_\alpha k_\beta)^\cdot , \tag{16}$$

so that its 4-acceleration is simply

$$a(k)_\beta = k^\mu \nabla_\mu k_\beta = \dot{k}_\beta . \tag{17}$$

The other optical scalars of interest are the expansion

$$\theta = \frac{1}{2} k^\alpha_{;\alpha} = \frac{1}{2} \eta^{\alpha\beta} k_{\beta,\alpha} = \frac{1}{2} k^\alpha_{,\alpha} , \tag{18}$$

the vorticity

$$\omega^2 = \frac{1}{2} k_{[\alpha;\beta]} k^{\alpha;\beta} = \frac{1}{2} k_{[\beta,\alpha]} \left(\eta^{\alpha\mu} \eta^{\beta\nu} k_{\mu,\nu} - 2\epsilon H \dot{k}^\alpha k^\beta \right) , \tag{19}$$

and the shear, implicitly defined by

$$\theta^2 + |\sigma|^2 = \frac{1}{2} k_{(\alpha;\beta)} k^{\alpha;\beta} = \frac{1}{2} k_{(\beta,\alpha)} \eta^{\alpha\mu} \eta^{\beta\nu} k_{\mu,\nu} - \frac{1}{2} \epsilon H ||\dot{\mathbf{k}}||^2 . \tag{20}$$

2.2. First result: \mathbf{k} be geodesic

The third order field equations (11) imply that \mathbf{k} be geodesic. Then it can be normalized so that $\dot{\mathbf{k}} = 0$. The optical scalars (19) and (20) thus become

$$\begin{aligned}
\omega^2 &= \frac{1}{2} \eta^{\alpha\mu} \eta^{\beta\nu} k_{[\beta,\alpha]} k_{\mu,\nu} , \\
\theta^2 + |\sigma|^2 &= \frac{1}{2} \eta^{\alpha\mu} \eta^{\beta\nu} k_{(\beta,\alpha)} k_{\mu,\nu} .
\end{aligned} \tag{21}$$

The second order Ricci tensor (12) simplifies to

$$R_{\alpha\beta} = 2H\mathcal{D}k_\alpha k_\beta , \quad \mathcal{D} = \ddot{H} + 2\theta\dot{H} + 4H\omega^2 , \tag{22}$$

leading to the condition $\mathcal{D} = 0$, which gives the following equation for H

$$0 = \ddot{H} + 2\theta\dot{H} + 4H\omega^2 . \tag{23}$$

Finally, the first order Ricci tensor (14)–(15) becomes

$$R_{\alpha\beta} = \eta^{\lambda\gamma} H_{,\lambda\gamma} k_{\alpha} k_{\beta} + 2k_{(\alpha} B_{\beta)} - 2 \left[(\dot{H} + 2\theta H) k_{(\alpha,\beta)} - \eta^{\lambda\gamma} H k_{\alpha,\gamma} k_{\beta,\lambda} \right] , \quad (24)$$

with

$$B_{\beta} = -(\dot{H} + 2\theta H)_{,\beta} + \eta^{\lambda\gamma} (2H_{,\lambda} k_{\beta,\gamma} + H k_{\beta,\gamma\lambda}) . \quad (25)$$

The vector \mathbf{k} is an eigenvalue of the Ricci tensor, i.e.

$$R_{\alpha\sigma} k^{\sigma} = (B_{\sigma} k^{\sigma}) k_{\alpha} . \quad (26)$$

It proves easier to handle with the remaining set of first order field equations by specifying a general field of real null direction in Minkowski space together with an adapted tetrad frame, then setting to zero each individual frame component of the first order Ricci tensor.

2.3. Simplified tetrad procedure

Following [5,6] introduce the set of null coordinates in Minkowski space $(u, v, \zeta, \bar{\zeta})$, which are related to the standard Cartesian coordinates (t, x, y, z) by

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(t - z) , & v &= \frac{1}{\sqrt{2}}(t + z) , \\ \zeta &= \frac{1}{\sqrt{2}}(x + iy) , & \bar{\zeta} &= \frac{1}{\sqrt{2}}(x - iy) . \end{aligned} \quad (27)$$

The metric (1) becomes

$$ds^2 = 2(d\zeta d\bar{\zeta} - dudv) - 2\epsilon H k_{\alpha} k_{\beta} dx^{\alpha} dx^{\beta} . \quad (28)$$

A general field of real null directions in Minkowski space is given by

$$k = -[du + Y\bar{Y}dv + \bar{Y}d\zeta + Yd\bar{\zeta}] , \quad \mathbf{k} = Y\bar{Y}\partial_u + \partial_v - Y\partial_{\zeta} - \bar{Y}\partial_{\bar{\zeta}} , \quad (29)$$

where Y is an arbitrary complex function of coordinates. In fact the independent components of \mathbf{k} reduce to two real functions of the coordinates, due to the two conditions 1) \mathbf{k} forms a lightlike world line and 2) \mathbf{k} has an arbitrary parametrization. In Eq. (29) these two real functions of the coordinates collapsed in a single complex function Y , namely $\mathbf{k} = \mathbf{k}(Y, \bar{Y})$.

We introduce the following frame

$$\omega^1 = d\zeta + Ydv , \quad \omega^2 = d\bar{\zeta} + \bar{Y}dv , \quad \omega^3 = -k , \quad \omega^4 = dv + \epsilon H \omega^3 , \quad (30)$$

so that

$$ds^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4 . \quad (31)$$

The dual frame is

$$\mathbf{e}_1 = \partial_{\zeta} - \bar{Y}\partial_u , \quad \mathbf{e}_2 = \partial_{\bar{\zeta}} - Y\partial_u , \quad \mathbf{e}_3 = \partial_u - \epsilon H \mathbf{k} , \quad \mathbf{e}_4 = \mathbf{k} . \quad (32)$$

6 *Donato Bini, Andrea Geralico and Roy P. Kerr*

The connection coefficients are given by

$$\Gamma_{cab} = -e_c^\mu e_{a\mu;\nu} e_b^\nu . \quad (33)$$

Note that $\omega_\mu^1 = -k_{\mu,\bar{Y}}$ and $\omega_\mu^2 = -k_{\mu,Y}$, trivially implying $\omega^1(\mathbf{k}) = 0 = \omega^2(\mathbf{k})$, because

$$\mathbf{k} \cdot \omega^1 = \eta^{\alpha\beta} k_\alpha \omega_\beta^1 = -\eta^{\alpha\beta} k_\alpha k_{\beta,\bar{Y}} = -k^\alpha k_{\alpha,\bar{Y}} = 0 . \quad (34)$$

Similarly $\mathbf{k} \cdot \omega^2 = 0$.

The derivative of \mathbf{k} is quite simple

$$k_{\mu,\nu} = k_{\mu,\bar{Y}} \bar{Y}_{,\nu} + k_{\mu,Y} Y_{,\nu} = -\omega_\mu^1 \bar{Y}_{,\nu} - \omega_\mu^2 Y_{,\nu} . \quad (35)$$

Next introduce the following standard notation for the directional derivatives along the frame vectors

$$\begin{aligned} D &\equiv \nabla_{\mathbf{k}} = \partial_v + Y \bar{Y} \partial_u - Y \partial_\zeta - \bar{Y} \partial_{\bar{\zeta}} , \\ \Delta &\equiv \nabla_{e_3} = \partial_u - \epsilon H D , \\ \delta &\equiv \nabla_{e_1} = \partial_\zeta - \bar{Y} \partial_u . \end{aligned} \quad (36)$$

The geodesic curvature κ , complex expansion ρ and shear σ of the null congruence \mathbf{k} are given by

$$\begin{aligned} \kappa &\equiv -\Gamma_{414} = -k^\alpha D e_{1\alpha} = D \bar{Y} , \\ \rho &\equiv -\Gamma_{412} = -k^\alpha \bar{\delta} e_{1\alpha} = \bar{\delta} \bar{Y} , \\ \sigma &\equiv -\Gamma_{411} = -k^\alpha \delta e_{1\alpha} = \delta \bar{Y} , \end{aligned} \quad (37)$$

respectively. It is also useful to introduce the quantity

$$\tau \equiv -\Gamma_{413} = -k^\alpha \Delta e_{1\alpha} = \partial_u \bar{Y} . \quad (38)$$

If the principal null vector \mathbf{k} is geodesic, then $\kappa = 0$, i.e.

$$0 = D \bar{Y} = \bar{Y}_{,v} + Y \bar{Y} \bar{Y}_{,u} - Y \bar{Y}_{,\zeta} - \bar{Y} \bar{Y}_{,\bar{\zeta}} . \quad (39)$$

If it is also shearfree, then $\sigma = 0$, i.e.

$$0 = \delta \bar{Y} = \bar{Y}_{,\zeta} - \bar{Y} \bar{Y}_{,u} , \quad \rightarrow \quad (\text{c.c.}) \quad 0 = Y_{,\bar{\zeta}} - Y Y_{,u} , \quad (40)$$

where “c.c.” stands for “complex conjugate.” Substituting it into Eq. (39) then yields

$$0 = \bar{Y}_{,v} - \bar{Y} \bar{Y}_{,\bar{\zeta}} , \quad \rightarrow \quad (\text{c.c.}) \quad 0 = Y_{,v} - Y Y_{,\zeta} . \quad (41)$$

The conditions (40) and (41) thus give

$$Y_{,\bar{\zeta}} = Y Y_{,u} , \quad Y_{,v} = Y Y_{,\zeta} , \quad (42)$$

whence if \square_0 is the flat-space wave operator, then

$$\square_0 Y \equiv \eta^{\alpha\beta} Y_{,\alpha\beta} = 2Y_{,\bar{\zeta}\zeta} - 2Y_{,uv} = (Y^2)_{,u\zeta} - (Y^2)_{,\zeta u} = 0 , \quad (43)$$

and therefore Y is a solution of the wave equation in Minkowski space whenever the congruence is geodesic and shearfree. They also show that the congruence \mathbf{k} must satisfy the Kerr Theorem, i.e. Y is a root of an analytic equation

$$0 = F(Y, \bar{\zeta}Y + u, vY + \zeta) , \quad (44)$$

where F is an arbitrary function analytic in the three complex variables Y , $\bar{\zeta}Y + u$ and $vY + \zeta$.

2.4. Completion of the solution

In terms of the connection coefficients previously introduced the optical scalars write as

$$\theta = -\frac{1}{2}(\rho + \bar{\rho}) , \quad \omega^2 = -\frac{1}{4}(\rho - \bar{\rho})^2 , \quad (45)$$

so that the single equation (23) coming from the vanishing of second order Ricci tensor reads

$$0 = \ddot{H} - (\rho + \bar{\rho})\dot{H} - (\rho - \bar{\rho})^2 H . \quad (46)$$

The nonvanishing relevant frame components of the first order Ricci tensor (24) are given by

$$R_{11} = 2\sigma[\dot{H} - (\bar{\rho} - \rho)H] , \quad (47a)$$

$$R_{12} = (\rho + \bar{\rho})\dot{H} - (\rho^2 + \bar{\rho}^2 - 2\sigma\bar{\sigma})H , \quad (47b)$$

$$R_{13} = \delta\dot{H} + (\rho - \bar{\rho})\delta H + 2\sigma\bar{\delta}H - \tau\dot{H} - (\delta\bar{\rho} + 2\bar{\tau}\sigma + 2\tau\rho - \bar{\delta}\sigma)H , \quad (47c)$$

$$R_{33} = 2[\delta\bar{\delta}H - (\rho_{,u} + \bar{\rho}_{,u})H - \tau\bar{\delta}H - \bar{\tau}\delta H - \rho H_{,u}] , \quad (47d)$$

$$R_{34} = \ddot{H} - (\rho + \bar{\rho})\dot{H} - (\rho - \bar{\rho})^2 H , \quad (47e)$$

since R_{22} and R_{23} are c.c. of R_{11} and R_{13} respectively. The identities

$$\begin{aligned} \bar{\delta}\tau &= \rho_{,u} + \tau\bar{\tau} , & \delta\tau &= \sigma_{,u} + \tau^2 , \\ \delta\rho &= \bar{\delta}\sigma + \tau(\rho - \bar{\rho}) , & D\rho &= \sigma\bar{\sigma} + \rho^2 , \\ D\tau &= \bar{\tau}\sigma + \tau\rho , & D\sigma &= \sigma(\rho + \bar{\rho}) , \end{aligned} \quad (48)$$

as well as the commutation relations

$$\begin{aligned} \partial_u D - D\partial_u &= -\bar{\tau}\delta - \tau\bar{\delta} , & \delta D - D\delta &= -\bar{\rho}\delta - \sigma\bar{\delta} , \\ \delta\partial_u - \partial_u\delta &= \tau\partial_u , & \bar{\delta}\delta - \delta\bar{\delta} &= -(\rho - \bar{\rho})\partial_u , \end{aligned} \quad (49)$$

have been used here to simplify the expressions involving frame derivatives of H . Setting to zero each component of Eqs. (47a)–(47e) gives a set of first order equations. Note that the condition coming from Eq. (47e) is the same as Eq. (46).

Equation (47a) implies $\sigma = 0$, i.e. the congruence \mathbf{k} must be shearfree. The remaining first order equations thus simplify as

$$0 = (\rho + \bar{\rho})\dot{H} - (\rho^2 + \bar{\rho}^2)H , \quad (50a)$$

$$0 = \delta\dot{H} + (\rho - \bar{\rho})\delta H - \tau\dot{H} - (\delta\bar{\rho} + 2\tau\rho)H , \quad (50b)$$

$$0 = \delta\bar{\delta}H - (\rho_{,u} + \bar{\rho}_{,u})H - \tau\bar{\delta}H - \bar{\tau}\delta H - \rho H_{,u} , \quad (50c)$$

and the identities (48) become

$$\begin{aligned} \bar{\delta}\tau &= \rho_{,u} + \tau\bar{\tau} , & \delta\tau &= \tau^2 , \\ \delta\rho &= \tau(\rho - \bar{\rho}) , & \dot{\rho} &= \rho^2 , \\ \dot{\tau} &= \tau\rho . \end{aligned} \quad (51)$$

Taking the δ derivative of Eq. (50a) together with Eq. (50b) gives rise to the following compatibility condition

$$\rho(\rho + \bar{\rho})\delta H = [\tau\rho(\rho + \bar{\rho}) + \tau\bar{\rho}(\rho + 3\bar{\rho}) + \rho\delta\bar{\rho}]H . \quad (52)$$

Take the complex conjugate of this equation and then its δ derivative; Eq. (50c) thus gives rise to a second compatibility condition

$$(\rho + \bar{\rho})H_{,u} = \left[3 \left(\frac{\rho\bar{\tau}}{\bar{\rho}^2}\delta\bar{\rho} + \frac{\bar{\rho}\tau}{\rho^2}\delta\rho \right) + (\bar{\rho} + 3\rho)\frac{\bar{\rho}_{,u}}{\bar{\rho}} + (\rho - 3\bar{\rho})\frac{\rho_{,u}}{\rho} + 6\frac{\tau\bar{\tau}}{\bar{\rho}} \right] H . \quad (53)$$

By using Eq. (50a), Eq. (46) rewrites as

$$\ddot{H} = 2(\rho + \bar{\rho})\dot{H} - 2\rho\bar{\rho}H . \quad (54)$$

Let the complex expansion be nonzero, i.e. $\rho \neq 0$. It is easy to check that $\rho\bar{\rho}$ and $\rho + \bar{\rho}$ are particular solutions, and therefore the general solution is

$$H = \frac{1}{2}M(\rho + \bar{\rho}) + B\rho\bar{\rho} , \quad \dot{M} = \dot{B} = 0 , \quad (55)$$

where $M(Y, \bar{Y})$ and $B(Y, \bar{Y})$ are real functions of Y and \bar{Y} . Substituting the general solution (55) for H into Eq. (50a) one easily gets $B = 0$, by using the relation $\dot{\rho} = \rho^2$, so that

$$H = \frac{1}{2}M(\rho + \bar{\rho}) . \quad (56)$$

Substituting now this solution for H into Eq. (52) leads to

$$\delta M = \frac{3M}{\rho}\tau\bar{\rho} . \quad (57)$$

But $M = M(Y, \bar{Y})$, so that $\delta M = M_{,Y}\bar{\rho}$, implying that

$$M_{,Y} = \frac{3M}{\rho}\tau , \quad M_{,\bar{Y}} = \frac{3M}{\bar{\rho}}\bar{\tau} . \quad (58)$$

The second compatibility condition (53) then yields

$$\frac{\bar{\rho}}{\rho} \left(\bar{\delta}\tau - \frac{\tau}{\rho}\bar{\delta}\rho \right) - \text{c.c.} = 0 , \quad (59)$$

where the relation

$$M_{,u} = 3M\tau\bar{\tau} \left(\frac{1}{\rho} + \frac{1}{\bar{\rho}} \right) \quad (60)$$

has been used. Equation (59) is an additional equation for Y and \bar{Y} which we will discuss later.

Following the original work [5] we now introduce $P = (M/m)^{-1/3}$, where m is a real constant. The first equation of (58) thus becomes

$$P^{-1}P_{,Y} = -\frac{\tau}{\rho} . \quad (61)$$

By taking δ of both sides we then find

$$-\bar{\rho}P^{-2}(P_{,Y})^2 + \bar{\rho}P^{-1}P_{,YY} = -\frac{\tau^2}{\rho^2}\bar{\rho} = -\bar{\rho}P^{-2}(P_{,Y})^2 , \quad (62)$$

since $\delta P = \bar{\rho}P_{,Y}$ and $\delta P_{,Y} = \bar{\rho}P_{,YY}$, and the identities (51) have been used to replace $\delta\rho$ and $\delta\tau$ on the RHS. Equation (62) thus implies $P_{,YY} = 0$, whose solution is

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c , \quad (63)$$

where p and c are real constants and q is a complex constant.

Let us turn to the remaining compatibility condition (59). First note that it can be equivalently rewritten as

$$\bar{\rho}\bar{\delta} \left(\frac{\tau}{\rho} \right) - \text{c.c.} = 0 . \quad (64)$$

By using Eq. (61) we have

$$\bar{\rho}\bar{\delta} \left(\frac{\tau}{\rho} \right) = \rho\bar{\rho}P^{-1}[P^{-1}P_{,Y}P_{,\bar{Y}} - P_{,Y\bar{Y}}] . \quad (65)$$

Take the complex conjugate of this expression taking into account that P is real; substituting then into Eq. (64) we find that it is identically satisfied.

Finally, taking the exterior derivative of Y gives

$$\begin{aligned} dY &= \delta Y\omega^1 + Y_{,u}\omega^3 = P^{-1}\bar{\rho}[P\omega^1 - P_{,\bar{Y}}\omega^3] \\ &= P^{-1}\bar{\rho}[(qY + c)(d\zeta + Ydv) - (pY + \bar{q})(du + Yd\bar{\zeta})] , \end{aligned} \quad (66)$$

whose general solution is

$$0 = F \equiv \phi(Y) + (qY + c)(\zeta + Yv) - (pY + \bar{q})(u + Y\bar{\zeta}) , \quad (67)$$

according to Eq. (44), with ϕ an arbitrary analytic function of the complex variable Y . In fact, differentiating Eq. (67) leads to

$$F_{,Y}dY = dF = F_{,\alpha}dx^\alpha = (qY + c)(d\zeta + Ydv) - (pY + \bar{q})(du + Yd\bar{\zeta}) . \quad (68)$$

Furthermore, taking the δ derivative of F , i.e.

$$\bar{\rho}F_{,Y} = \delta F = (\partial_\zeta - \bar{Y}\partial_u)F = P , \quad (69)$$

10 *Donato Bini, Andrea Geralico and Roy P. Kerr*

implies that the complex expansion of the null vector \mathbf{k} is given by

$$\bar{\rho} = PF_{,Y}^{-1} . \quad (70)$$

Equation (66) then immediately follows.

Summarizing, the solution is given by

$$ds^2 = 2(d\zeta d\bar{\zeta} - dudv) - \frac{m}{P^3}(\rho + \bar{\rho})[du + Y\bar{Y}dv + \bar{Y}d\zeta + Yd\bar{\zeta}]^2 , \quad (71)$$

with

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c , \quad \bar{\rho} = PF_{,Y}^{-1} . \quad (72)$$

The main properties of such a family of solutions are listed below (see e.g. [7]):

1. They are all algebraically special, with \mathbf{k} shearfree and geodesic.
2. They all admit at least a one-parameter group of motions with Killing vector

$$\xi = c\partial_u + p\partial_v + \bar{q}\partial_\zeta + q\partial_{\bar{\zeta}} , \quad (73)$$

which is simultaneously a Killing vector of flat spacetime. The solutions can be simplified by performing a Lorentz transformation. One can thus assume that if

- a) $\eta_{\alpha\beta}\xi^\alpha\xi^\beta < 0$, then $P = (1 + Y\bar{Y})/\sqrt{2}$, i.e. with ξ pointing along the $u + v$ (or t) direction ($p = c = 1/\sqrt{2}$, $q = 0$);
 - b) $\eta_{\alpha\beta}\xi^\alpha\xi^\beta > 0$, then $P = (1 - Y\bar{Y})/\sqrt{2}$, i.e. with ξ pointing along the $v - u$ (or z) direction ($-p = c = 1/\sqrt{2}$, $q = 0$);
 - c) $\eta_{\alpha\beta}\xi^\alpha\xi^\beta = 0$, then $P = 1$, i.e. with ξ pointing along the u direction ($p = q = 0$, $c = 1$).
3. For a timelike Killing vector ξ , the particular case $\phi = -iaY$, with $m = \mathcal{M}$, leads to the Kerr solution (1)–(6), once written in Kerr-Schild coordinates.

3. Concluding remarks

We have presented an alternative derivation of Kerr solution by treating Kerr-Schild metrics as *exact linear perturbations* of Minkowski spacetime. In fact they have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field \mathbf{k} multiplied by a scalar function H .

In the case of Kerr solution the vector \mathbf{k} is geodesic and shearfree and it is independent of the mass parameter \mathcal{M} , which enters instead the definition of H linearly. This linearity property allows one to solve the field equations order by order in powers of H in complete generality, i.e. without any assumption on the null congruence \mathbf{k} . The Ricci tensor turns out to consist of three different contributions. Third order equations all imply that \mathbf{k} must be geodesic; it must be also shearfree as a consequence of first order equations, whereas the solution for H comes from second order equations too.

The present treatment can be generalized to include also the electromagnetic field, i.e. to the case of Kerr-Newman. In fact, even in the charged Kerr solution the congruence of k -lines depend only on the rotation parameter a and not on the mass \mathcal{M} or charge Q . Furthermore, the electromagnetic field is linear in Q and the metric is linear in \mathcal{M} and Q^2 , since the function H is obtained simply by replacing $\mathcal{M} \rightarrow \mathcal{M} - Q^2/(2r)$.

Acknowledgements

We thank Prof. R. Ruffini and ICRANet for support.

References

- [1] R. P. Kerr and A. Schild, A new class of vacuum solutions of the Einstein field equations, in *Atti del convegno sulla relatività generale; problemi dell'energia e onde gravitazionali*, ed. G. Barbera (Firenze, 1965) p. 173.
- [2] R. P. Kerr and A. Schild, *Proc. Symp. Appl. Math.* **17** (1965), 199.
- [3] R. P. Kerr, *Phys. Rev. Lett.* **11** (1963), 237.
- [4] J. N. Goldberg and R. K. Sachs, *Acta Phys. Polon.* **22 Suppl.** (1962), 13.
- [5] G. Debney, R. P. Kerr and A. Schild, *J. Math. Phys.* **10** (1969), 1842.
- [6] R. P. Kerr and W. B. Wilson, *Gen. Rel. Grav.* **10** (1979), 273.
- [7] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed. (Cambridge Univ. Press, Cambridge, 2003).